A Sensitive Method for Measuring Complex Permittivity with a Microwave Resonator

G. ROUSSY AND M. FELDEN

Abstract—The proposed method was carried out to make easier dielectric studies on powders under controlled pressure when the frequency varies. A TE_{01n}-mode resonator is needed. The substance under test is contained in a tube of quartz, placed along the axis of a cylindrical cavity.

The mathematical formulation for the complex permittivity is given in rigorously accounting for the presence of the tube. Typical difficulties are discussed, and experimental results given.

The absolute precision of the measurement can be compared with that of the classical method (± 2 percent for ϵ' and ± 10 percent for ϵ''). The greatest error arises from an insufficiently precise determination of the geometrical and electrical characteristics of the quartz tube. This error is systematic, and thus it is possible to demonstrate the very small permittivity variations (± 0.3 percent for ϵ' and ± 5 percent for ϵ'').

Introduction

THE METHODS and equipment employed for microwave measurements of permittivity are frequently described in literature because of the various possible values of permittivity [1]. The real part ϵ' can vary from 1 to 10 000, and the loss factor ϵ''/ϵ' from 10^{-5} to 1. The nature of the material under investigation must be taken into account when choosing the possible methods. It is well known that for low loss or very high loss dielectric a resonator method [2] or a slotted line method [3] is used. When the losses are between the two, neither of these solutions is necessary a priori.

For numerous physical or chemical studies, the sensitivity of the measurement is essential. The demonstration of a variation of permittivity can indicate a chemical modification of the substance studied. From the study of permittivity variation as a function of frequency, we can calculate several molecular characteristics.

Most of the difficulties and inaccuracies of each method arise from the fact that the size of the sample is not exactly known. Frequently, an accompanying phenomena modifies it. When a liquid is tested, for example, it is very difficult to eliminate the bubbles which form on metallic surfaces, or if the cell is not completely filled, the free surface of the liquid is not plain because of the capillarity; in other cases, the presence of gas in the empty volume above the liquid can be troublesome.

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When the substance is a powder, it is difficult to make two samples in which the porosity and packing characteristics are identical. It is impossible to use the method of Von Hippel in optimal conditions of accuracy if we wish, in addition, to vary the frequency. In effect, the optimal size of the sample is a function of the frequency [3].

The object of the work described in this paper is to develop a cavity arrangement which would be free from the above-mentioned difficulties.

It is well known that high precision dielectric measurement of low loss material can be performed by means of a TE_{01n} cylindrical resonator, especially when a circular cylindrical solid rod sample, centered in the cavity, is used [4]. One advantage of this method is that it is possible to choose the frequency in a large range without diminishing the accuracy (from 9000 Mc/s to 10 000 Mc/s, for example). We can extend the range still more by using a single rod in different cavities. The substance, if not a solid, can be contained in a cylindrical quartz tube, as shown in Fig. 1. This arrangement has already been considered by Collie, Hasted, and Ritson [5], in the particular case of a capillary tube containing a polar liquid. They determined its refractive index when they knew its absorption coefficient.

The following analysis yields simultaneously the two components of the permittivity of most conventional dielectrics ($\epsilon'/\epsilon_0 \# 3$, $\epsilon''/\epsilon' \# 10^{-2}$). It strictly takes into account the presence of the tube, which may be of selected geometric dimensions.

Theory

The principle of calculating the permittivity of the substance from the resonance conditions is well known. We limit ourselves to citing our results (not found elsewhere in literature to our knowledge). The notations are defined at the end of the paper.

The field distribution is assumed at resonance to be the juxtaposition of three TE_{01n} distributions.

For $0 \leq r \leq c$ (medium 3):

$$\begin{cases} H_z = AJ_0(k_3 r) \sin(n\pi z/L) e^{j\omega t} \\ H_r = -(n\pi/Lk_3) AJ_1(k_3 r) \cos(n\pi z/L) e^{j\omega t} \\ E_\phi = -j(\omega\mu_0/k_3) AJ_1(k_3 r) \sin(n\pi z/L) e^{j\omega t}. \end{cases} \quad (1)$$

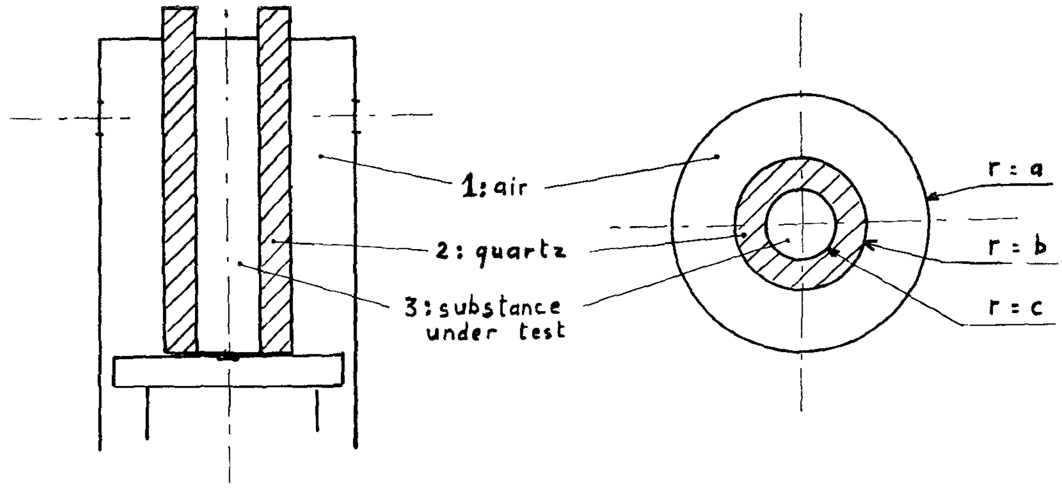

Fig. 1. Arrangement of the tube in the cavity.

For $c \leq r \leq b$ (medium 2):

$$\begin{cases} H_z = [BJ_0(k_2r) + CY_0(k_2r)] \sin (n\pi z/L) e^{j\omega t} \\ H_r = -(n\pi/Lk_2)[BJ_1(k_2r) + CY_1(k_2r)] \cos (n\pi z/L) e^{j\omega t} \\ E_\phi = -j(\omega\mu_0/k_2)[BJ_1(k_2r) + CY_1(k_2r)] \sin (n\pi z/L) e^{j\omega t}. \end{cases} \quad (2)$$

For $b \leq r \leq a$ (medium 1):

$$\begin{cases} H_z = [DJ_0(k_1r) + EY_0(k_1r)] \sin (n\pi z/L) e^{j\omega t} \\ H_r = -(n\pi/Lk_1)[DJ_1(k_1r) + EY_1(k_1r)] \cdot \cos (n\pi z/L) e^{j\omega t} \\ E_\phi = -j(\omega\mu_0/k_1)[DJ_1(k_1r) + EY_1(k_1r)] \cdot \sin (n\pi z/L) e^{j\omega t} \end{cases} \quad (3)$$

with:

$$\beta^2 = \epsilon_1'\mu_0\omega^2 - k_1^2 = \epsilon_2'\mu_0\omega^2 - k_2^2 = \epsilon_3'\mu_0\omega^2 - k_3^2 \quad (4)$$

$$\beta L = n\pi. \quad (5)$$

A, B, C, D, and E are constants of integration. They must satisfy the boundary conditions at the surfaces separating the media. After eliminating these constants, we obtain:

$$\frac{J_1(x)}{x \cdot J_0(x)} = \frac{muTQ - vPW}{muVQ - vPX} (1/nv). \quad (6)$$

The quantities P, Q, T, V, W, and X are given in the Appendix.

The stored energy in the media $i = 1$, 2, and 3 is:

$$W_i = \pi L \mu_0 A^2(\beta^2 + k_i^2) M_i / 4k_i^2. \quad (7)$$

The power loss in the curved metal surface and ends of the cavity is:

$$W_d = (\pi/4) A^2 \omega d \mu_0 \{ aLZ_0^2(u) + 2\beta^2[(M_1/k_1^2) + (M_2/k_2^2) + (M_3/k_3^2)] \}. \quad (8)$$

It follows the loss factor of the studied substance:

$$\epsilon_3''/\epsilon_3' = (W_1 + W_2 + W_3)/W_3Q - (W_d/W_3) - W_2\epsilon_2''/W_3\epsilon_2'. \quad (9)$$

The resonance curve is delineated at fixed frequency by variation of the resonator length. The Q factor can be obtained in terms of width of resonance at half-height:

$$\frac{1}{Q} = \beta^2 \frac{\sum_{i=1}^{3} (M_i/k_i^2)}{\sum_{i=1}^{3} M_i(\beta^2 + k_i^2)/k_i^2} \cdot \frac{\Delta L}{L}. \quad (10)$$

When the cavity is empty, the width of the resonance curve gives, in fact, the "effective" depth of current penetration:

$$d = \frac{a\beta_0^2 \Delta L_0}{L_0 k_0^2 + 2a\beta_0^2}. \quad (11)$$

We substitute the value of $1/Q$ and d in (9) to obtain the loss factor of the studied substance:

$$\frac{\epsilon_3''}{\epsilon_3'} = \frac{k_3^2}{(\beta^2 + k_3^2)LM_3} \left\{ \beta^2 \left(\frac{M_1}{k_1^2} + \frac{M_2}{k_2^2} + \frac{M_3}{k_3^2} \right) \Delta L - \frac{a\beta_0^2}{2a\beta_0^2 + L_0k_0^2} \cdot \left[aLZ_0^2(u) + 2\beta^2 \left(\frac{M_1}{k_1^2} + \frac{M_2}{k_2^2} + \frac{M_3}{k_3^2} \right) \right] \Delta L_0 - LM_2(\beta^2 + k_0^2)\epsilon_2''/\epsilon_2' k_2^2 \right\}. \quad (12)$$

Experimental

The simplest and most accurate way to observe the properties of the resonator is the transmission assembly (Fig. 2). It has been described several times in the litera-

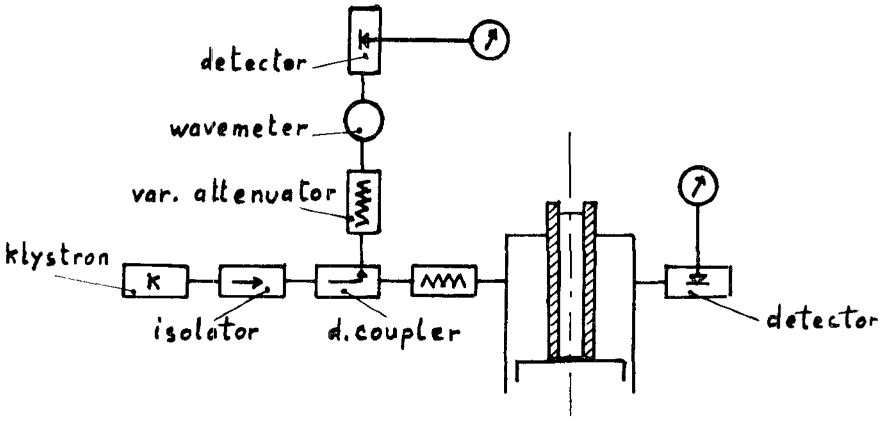

Fig. 2. Microwave assembling.

TABLE I
CHARACTERISTICS OF TUBES

No. of the tube	b_{mm}	c_{mm}	Δc_{mm}	ϵ_2'/ϵ_0
1	2.480 ± 0.005	2.10 ± 0.01	± 0.003	3.75
2	3.502 ± 0.005	2.49 ± 0.02	± 0.005	3.75
3	4.492 ± 0.005	3.50 ± 0.02	± 0.005	3.75

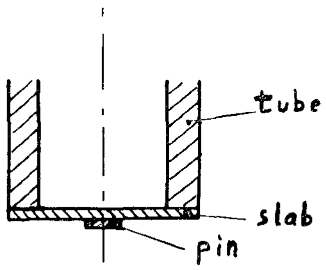

Fig. 3. Cross section of the tube.

ture [6]. The behavior of the cavity when its length varies has been studied elsewhere [7]. We will only cite here several points which seem essential.

The klystron frequency must be stabilized (fluctuations less than 30 kc/s). Care must be taken during the fabrication of the cavity because the position of the plunger must be known with high accuracy ($\pm 1\mu$). The cavity used is described by Roussy [8]. For frequencies between 9000 and 10 000 Mc/s, both resonances TE_{011} and TE_{012} are observed. The effects of coupling and that of the hole in the wall through which the vessel projects are considered in comparing d_1 and half d_2, or in comparing directly the results obtained for the two resonances TE_{011} and TE_{012}.

Different quartz tubes have been used. Their sizes are given in Table I.

The measure of the outside diameter (2b) is simple and accurate. However, we had to be satisfied with an approximate optical measure of the inside diameter. The fourth column of the table gives the probable tolerance. Tubes 2 and 3 were specially manufactured by Quartz et Silice, Paris, France. One of their ends was closed by a small slab as shown on Fig. 3. They were located centrally in the resonator by means of a pin (0.3 mm in diameter, 0.3 mm high), placed centrally in the face of the slab and engaging in a small hole in the face of the plunger. Tube 1 is a sample tube of A.60 Varian-R.M.N. spectrometer (part number 905.370).

Results and Discussion

In order to judge the accuracy of the method, we have considered substances whose permittivities are known. Results given in Table II were obtained with approximate values of permittivity and inside diameter of tube.

Important discrepancies are observed when the tube is changed. These are essentially due to the fact that inside diameter and tube permittivity are not known to sufficient accuracy. An error of $\pm 2/100$ mm on the inside diameter contributes an uncertainty, depending on the diameter and on the substance introduced, but which on the average is ± 7 percent of the result. Likewise an error of ± 1 percent on the permittivity allows an average error of ± 2 percent.

However, if the same experiment is repeated and, providing that series of measures is exploited with same values for the above-mentioned parameters, all of the results are within ± 0.3 percent for the real part and within ± 5 percent for the imaginary one. This high sensitivity is a great advantage.

In order to reduce the systematic error, correcting terms $\delta\epsilon_2'$ and δc which must be algebraically added to ϵ_2' and c have been calculated. The results reported in Table III are obtained after calibration.

It should be noted that, for low loss substances such as air, carbon tetrachloride, or benzene, this method is not very precise in absolute values. The dielectric size

TABLE II

Tube		Air		CCl_4		C_6H_6		C_2H_5Br	
		ϵ'/ϵ_0	ϵ''/ϵ'	ϵ'/ϵ_0	ϵ''/ϵ'	ϵ'/ϵ_0	ϵ''/ϵ'	ϵ'/ϵ_0	ϵ''/ϵ'
1	$n=1$	1.625	0.040	2.148	0.223	3.404	0.113	8.806	0.195
	$n=2$	—	0.137	2.138	0.127	3.474	0.060	8.773	0.196
2	$n=1$	1 092	0.014	2.344	—	3.280	0.108	8.28	0.251
	$n=2$	1.070	0.0045	2.330	0.0073	3.205	0.098	—	—
2	$n=1$	1.082	0.00071	2.317	0.0027	2.831	0.0045	—	—
	$n=2$	1.083	0.00070	2.316	0.0022	2.841	0.0048	—	—

TABLE III
DEFINITIVE RESULTS

Tube	Air		CCl_4		C_6H_6	
	ϵ'/ϵ_0	ϵ''/ϵ'	ϵ'/ϵ_0	ϵ''/ϵ'	ϵ'/ϵ_0	ϵ''/ϵ'
2	0.996	0.00048	2.178	0.0076	2.572	0.102
3	1.016	0.00047	2.242	0.0022	2.469	0.049

is too small, especially when Tube 1 is involved. For these liquids or gases it is obviously simpler to fill up the cavity.

On the other hand, for intermediate loss substances, the inside tube diameter must be chosen as the inverse function of the losses of the substance studied so that the Q-factor remains high ($\Delta L \leq 180\mu$).

CONCLUSIONS

A method has been presented for measuring the permittivity of various materials—including powder—of which the real permittvity is less than 8 and of which the loss factor is near 10^{-2}.

The dominant error is systematic because it is associated with the tube characteristics. The method appears to be convenient for carrying out studies that can be done with the same sample tube. In this case, the sensitivity is great since it reaches ± 0.3 percent and ± 5 percent for real and imaginary parts of permittivity. The application is simple, and it avoids some experimental difficulties.

APPENDIX

List of Principal Notations

Index 0 affects the empty cavity parameters.

Indexes 1, 2, and 3 affect, respectively, that of media, studied substance, and quartz and air, except when Bessel functions J_0, J_1, Y_0, and Y_1 are involved.

a = radius of the cavity.

b = outside radius of the tube.

c = inside radius of the tube.

d_n = sliding of the plunger between the TE_{01n} resonances when the cavity is empty and contains the sample.

L_0, L = resonance length for empty and loaded cavity.

β = phase coefficient in the composite circular wave guide.

ω = angular frequency.

μ_0 = permeability of free space or media 1, 2, and 3.

ϵ_0, ϵ_1^*, ϵ_2^*, ϵ_3^* = permittivities of free space, media 1, 2, and 3.

$$\epsilon_i^* = \epsilon_i' - j\epsilon_i''$$

ΔL_{0n}, ΔL_n = width of resonance at half-height for empty and loaded cavity.

The MKSA system of units and cylindrical coordinates r, ϕ, and z are used throughout.

Derivation of Symbols M_1, M_2, and M_3

It is convenient to lead:

$$x = k_3 c \qquad m = b/a$$

$$u = k_1 a \qquad n = c/b$$

$$v = k_2 b.$$

$$P = J_1(u)Y_1(mu) - J_1(mu)Y_1(u)$$

$$Q = J_1(u)Y_0(mu) - J_0(mu)Y_1(u)$$

$$R = J_0(u)Y_1(mu) - J_1(mu)Y_0(u)$$

$$S = J_0(u)Y_0(mu) - J_0(mu)Y_0(u)$$

$$T = J_1(v)Y_1(nv) - J_1(nv)Y_1(v)$$

$$V = J_1(v)Y_0(nv) - J_0(nv)Y_1(v)$$

$$W = J_0(v)Y_1(nv) - J_1(nv)Y_0(v)$$

$$X = J_0(v)Y_0(nv) - J_0(nv)Y_0(v)$$

$$Z_0(v) = (\pi nv/2)[J_0(x)W - (nv/x)J_1(x)X]$$

$$Z_1(v) = (\pi nv/2)[J_0(x)T - (nv/x)J_1(x)V]$$

$$Z_0(u) = (\pi mu/2)[Z_0(v)R - (mu/v)Z_1(v)S].$$

Then:

$$M_1 = a^2\{Z_0^2(u)\} - b^2\{(mu/v)^2 Z_1^2(v) - (2/v)Z_1(v)Z_0(v) + Z_0^2\}$$

$$M_2 = b^2\{Z_1^2(v) - (2/v)Z_1(v)Z_0(v) + Z_0^2(v)\} - c^2\{(nv/x)^2 J_1^2(x) - (2/x)J_1(x)J_0(x) + J_0^2(x)\}$$

$$M_3 = c^2\{J_1^2(x) - (2/x)J_1(x)J_0(x) + J_0^2(x)\}.$$

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The Cutoff Wavelength of the TE_{10} Mode in Ridged Rectangular Waveguide of Any Aspect Ratio

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Abstract—**Design curves for ridged rectangular waveguides of the usual aspect ratio of 0.45 were published by Hopfer following earlier work carried out by Cohn and Marcuvitz.**

This paper has been written to extend the design data to ridged rectangular waveguides of any aspect ratio. An analysis of the error introduced by proximity effects has shown these to be of the order of a few percent.

I. INTRODUCTION

PUBLISHED design information on ridged rectangular waveguides may be found in the *Waveguide Handbook* [1] and in papers by Cohn [2] and Hopfer [3]. Hopfer gives design curves for the cutoff wavelength λ_c of the TE_{10} mode in single and double-ridged rectangular waveguide of the usual aspect ratio of 0.45. For other aspect ratios he provides correction curves, which, he states, are essentially a first-order correction on the value of the cutoff wavelength at the aspect ratio of 0.45.

The purpose of this paper is:

1) to repeat Hopfer's work for the single and double ridge in rectangular waveguide of an aspect ratio of 0.45, and to present the design data in tabulated form to permit accurate interpolation between computed values;
2) to extend the design data to cover aspect ratios other than 0.45 while maintaining the same accuracy as for the above case, and to present this information graphically as a correction to be applied to the values for an aspect ratio of 0.45; and
3) to investigate the errors which are introduced by such proximity effects as narrow ridges and well-formed ridges near the side wall of the waveguide.

II. CUTOFF CONDITION OF THE TE_{10} MODE IN RECTANGULAR WAVEGUIDE

Consider the rectangular waveguide shown in Fig. 1 where a and b are the transverse dimensions and b/a is the aspect ratio. The transmission of the TE_{10} wave within the metal walls may be represented by two converging TEM waves of free space wavelength λ_0 and angle 2θ between wave fronts. From the geometry of the system it follows that

$$\cos\theta = \frac{\lambda_0}{2a}. \tag{1}$$

The TE_{10} wave is cut off when θ is zero and when the TEM wave is traveling normally to the axis of the waveguide. This condition occurs at the cutoff wavelength λ_c given by

$$\lambda_c = 2a. \tag{2}$$

The cutoff wavelength could have been obtained in another way, by considering the rectangular waveguide

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